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THE CENTROID AND INERTIA TENSOR
FOR A SPHERICAL TRIANGLE

by

John E. Brock

1 November 1974

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NAVAL POSTGRADUATE SCHOOL
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THE CENTROID AND INERTIA TENSOR
FOR A SPHERICAL TRIANGLE

by

John E. Brock
Professor of Mechanical Engineering, NPS

The list of formulas for inertia tensors given by E. A. Milne in his textbook *Vectorial Mechanics* (Interscience, 1948) includes many of the fundamental mass distributions which are useful in practice. About twenty years ago the writer thought it would be interesting to add to this list a formula for the case of uniform mass distribution on the surface of a general spherical triangle, but other occupations have prevented his completing this task until quite recently.

A very brief note, giving the formula for the inertia tensor and also the formula locating the mass center, has been submitted for publication in a standard journal so as to make the results widely available. However, even though the derivations are straightforward, the viewpoint and enough of the details are sufficiently complicated to warrant their preservation and it is the purpose of this brief monograph to do this.

We are concerned with a uniform mass distribution over the spherical triangle T:ABC (cf. Figure 1) which lies on the surface of a sphere having radius r and center O. Triangle T is specified by vectors $\bar{a}$, $\bar{b}$, and $\bar{c}$ such that $r\bar{a} = \overline{OA}$, $r\bar{b} = \overline{OB}$, and $r\bar{c} = \overline{OC}$. We are also given either M, the total mass, or m, the mass per unit area. Without ultimate loss of generality we take r = 1 and require that each of the angles

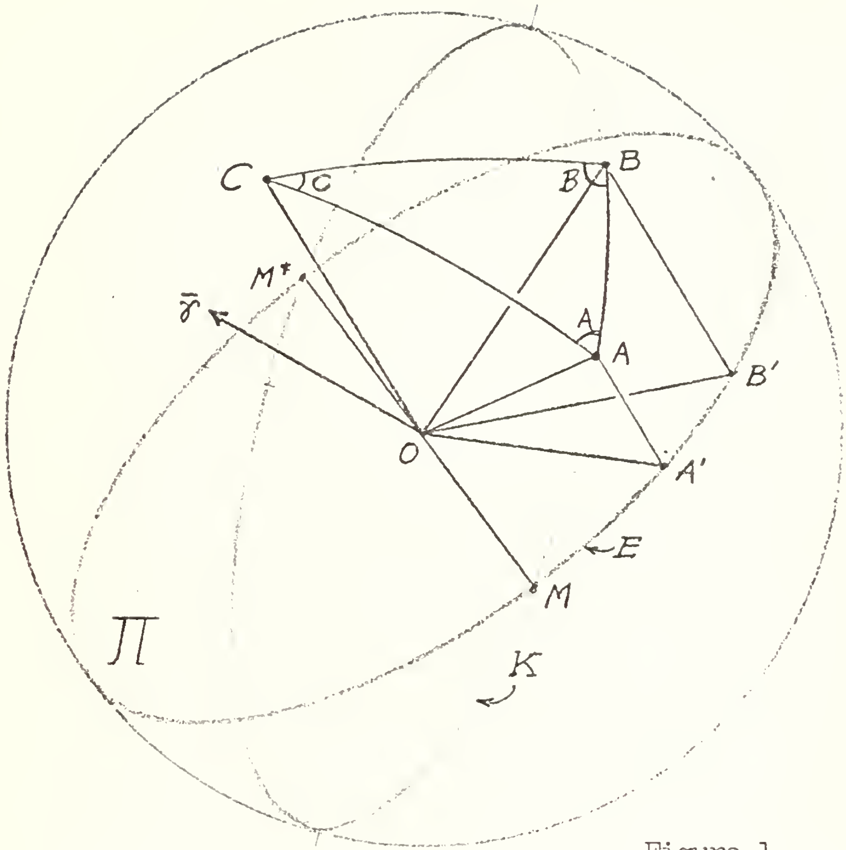

Figure 1

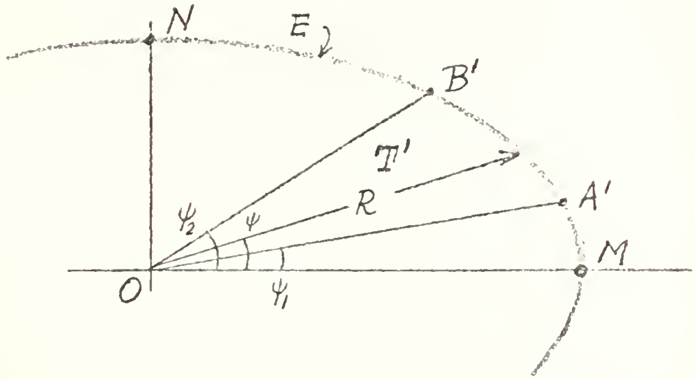

Figure 2

$\hat{A}$ = angle(BOC), $\hat{B}$ = angle(COA), and $\hat{C}$ = angle(AOB), respectively, not exceed $\pi/2$. This assures a projection property we will use and also that $\bar{a}\cdot\bar{b}\times\bar{c} > 0$. Let the letters A, B, and C also denote the vertex angles of T at A, B, and C. respectively. Also, introduce the following notations:

$$\sigma = \bar{a}\cdot\bar{b}\times\bar{c}$$

$$\bar{\alpha} = \bar{b}\times\bar{c}/\sigma, \quad \bar{\beta} = \bar{c}\times\bar{a}/\sigma, \quad \bar{\gamma} = \bar{a}\times\bar{b}/\sigma$$

$$\lambda = \bar{b}\cdot\bar{c}, \quad \mu = \bar{c}\cdot\bar{a}, \quad \nu = \bar{a}\cdot\bar{b}$$

$$\zeta = \sqrt{1-\lambda^2}, \quad \eta = \sqrt{1-\mu^2}, \quad \theta = \sqrt{1-\nu^2}$$

Note that

$$|\bar{\alpha}| = \zeta/\sigma, \quad |\bar{\beta}| = \eta/\sigma, \quad |\bar{\gamma}| = \theta/\sigma$$

and also recall the elementary results

$$\sigma^2 = 1 - \lambda^2 - \mu^2 - \nu^2 + 2\lambda\mu\nu$$

$$\mathcal{U} = \bar{a}\bar{\alpha} + \bar{b}\bar{\beta} + \bar{c}\bar{\gamma}$$

(We use script letters to denote dyadics; $\mathcal{U}$ denotes the unit dyadic.)

Our general procedure in determining a desired tensor $\mathcal{A}$ or vector $\bar{v}$ will be by use of the formulas

$$\mathcal{A} = \mathcal{A}\cdot\mathcal{U} = (\mathcal{A}\cdot\bar{a})\bar{\alpha} + (\mathcal{A}\cdot\bar{b})\bar{\beta} + (\mathcal{A}\cdot\bar{c})\bar{\gamma}$$

$$\bar{v} = \bar{v}\cdot\mathcal{U} = (\bar{v}\cdot\bar{a})\bar{\alpha} + (\bar{v}\cdot\bar{b})\bar{\beta} + (\bar{v}\cdot\bar{c})\bar{\gamma}$$

and to determine the dot-product $\mathcal{A}\cdot\bar{c}$ or $\bar{v}\cdot\bar{c}$. By a cyclical interchange of parameters, the other dot-products may easily be determined, thus establishing the desired result.

In this procedure we will deal with the projection of T into a plane figure T' lying in the plane Π which passes through O perpendicular to $\bar{c}$. The dyadic which performs this operation is

$$\mathcal{P} = \mathcal{U} - \bar{c}\bar{c}$$

Arc AB is part of the great circle K whose normal is in the direction of $\bar{\gamma}$. It projects into the ellipse E in Π. The "node" M is at unit distance from O and in the direction of

$$\bar{p} = \bar{c}\times(\bar{a}\times\bar{b}) = \lambda\bar{a} - \mu\bar{b} = p\bar{m}$$

where $\bar{m}$ is the unit vector $\overline{OM}$ and the magnitude of $\bar{p}$ is

$$p = \sqrt{\lambda^2 + \mu^2 - 2\lambda\mu\nu}$$

We also note that

$$\theta^2 = p^2 + \sigma^2$$

Let A' and B' be the projections of A and B, respectively, and let the unit vectors $\bar{a}'$ along $\overline{OA'}$ and $\bar{b}'$ along $\overline{OB'}$ be defined by the relations

$$\overline{OA'} = P\cdot\bar{a} = \bar{a} - \mu\bar{c} = \eta\bar{a}', \quad \overline{OB'} = P\cdot\bar{b} = \bar{b} - \lambda\bar{c} = \zeta\bar{b}'$$

Although we note that

$$\sin C = \bar{c}\cdot\bar{a}'\times\bar{b}' = \sigma/\eta\zeta$$

we can also use

$$\cos C = \bar{a}'\cdot\bar{b}' = (\nu-\lambda\mu)/\eta\zeta$$

to determine C without ambiguity. Similar formulas determine vertex angles A and B.

We will let P be a general point of T, let $\bar{r}$ denote the (unit) vector $\overline{OP}$, and let ϕ denote the angle COP. If dS is an areal element of T, its projection on Π is the areal element

$$dS' = dS \cos\phi$$

of the figure T' which is bounded by straight lines OA' and OB' and by a portion of the ellipse E whose semi-major axis is unity and whose semi-minor axis is $\bar{c}\cdot\bar{\gamma}\sigma/\theta = \sigma/\theta$. Accordingly, by measuring a polar angle ψ from "nodal" line OM, the polar equation of E may be determined to be

$$R^2 = \sigma^2/(\sigma^2 + p^2\sin^2\psi)$$

It will be useful to establish the relations summarized in Table 1, below. (Cf. the drawing of figure T' shown in Figure 2.) In particular we calculate

$$\cos\psi_1 = \bar{m}\cdot\bar{a}'; \ \cos\psi_2 = \bar{m}\cdot\bar{b}'; \ \cos(MOA) = \bar{m}\cdot\bar{a}; \ \cos(MOB) = \bar{m}\cdot\bar{b}$$

TABLE 1 Angles and their functions

Symbol	Description	Sine	Cosine
A	Vertex at A	$\sigma/\eta\theta$	$(\lambda-\mu\nu)/\eta\theta$
B	Vertex at B	$\sigma/\theta\zeta$	$(\mu-\nu\lambda)/\theta\zeta$
C	Vertex at C	$\sigma/\zeta\eta$	$(\nu-\lambda\mu)/\zeta\eta$
ψ_1	MOA'	$\mu\sigma/p\eta$	$(\lambda-\mu\nu)/p\eta$
ψ_2	MOB'	$\lambda\sigma/p\zeta$	$-(\mu-\nu\lambda)/p\zeta$
–	MOA	$\mu\theta/p$	$(\lambda-\mu\nu)/p$
–	MOB	$\lambda\theta/p$	$-(\mu-\nu\lambda)/p$

The area of T is

$$\begin{aligned} S = \int_T dS &= \int_{T'} \sec\phi \; dS' = \int_{\psi_1}^{\psi_2}\int_0^R \frac{\rho \, d\rho}{\sqrt{1-\rho^2}} \, d\theta \\ &= \int_{\psi_1}^{\psi_2} [1 - \sin\psi/\sqrt{(\theta/p)^2-\cos^2\psi}] \, d\psi \\ &= [\psi - \arccos(\frac{p\cos\psi}{\theta})]_{\psi_1}^{\psi_2} \\ &= (\psi_2-\psi_1) - \arccos(-\cos A) + \arccos(\cos B) \\ &= C - (\pi - B) + A = A + B + C - \pi = e \end{aligned}$$

where we have used the symbol e to denote the "spherical excess." This result is, of course, very well known and the purpose of making the calculation here is simply to indicate the method of integration which will be used to establish other useful results.

We will next locate the mass center G of the distribution by evaluating the vector

$$\overline{OG} = \bar{g} = \frac{1}{S}\int_T \bar{r}\, dS = (\bar{g}\cdot\bar{a})\bar{\alpha} + (\bar{g}\cdot\bar{b})\bar{\beta} + (\bar{g}\cdot\bar{c})\bar{\gamma}$$

Therefore

$$S\bar{c}\cdot\bar{g} = \int_T \bar{c}\cdot\bar{r}\, dS = \int_T \cos\phi\, dS = \int_{T'} dS' = S'$$

$$= \int_{\psi_1}^{\psi_2}\int_0^R \rho\, d\rho\, d\psi = \frac{1}{2}\int_{\psi_1}^{\psi_2} R^2\, d\psi = \frac{\sigma^2}{2}\int_{\psi_1}^{\psi_2} \frac{d\psi}{\theta^2\sin^2\psi + \sigma^2\cos^2\psi}$$

$$= (\sigma/2\theta)\{\arctan[(\theta/\sigma)\tan\psi]_{\psi_1}^{\psi_2}$$

$$= (\sigma/2\theta)[\mathrm{angle}(MOB) - \mathrm{angle}(MOA)] = \sigma\hat{C}/2\theta$$

Similar results may be obtained for $\bar{a}\ \bar{g}$ and $\bar{b}\ \bar{g}$, and upon removing the restriction $r = 1$, we obtain

$$\bar{g} = (r/2e)[\frac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}\hat{C} + (\sim) + (\sim)^2]$$

where the symbol $(\sim)$ indicates that the term is to be obtained by the cyclic interchange $\bar{a} \to \bar{b} \to \bar{c}$ and $A \to B \to C$.

Again temporarily assuming $r = 1$, we note that the definition of the tensor of inertia, at O, of this mass distribution, is

$$I_o = m\int_T (U - \bar{r}\bar{r})\, dS$$

and it is clear that only the second term imposes any difficulty. Write

$$J = \int_T \bar{r}\bar{r}\, dS$$

Now

$$\bar{r} = U\cdot\bar{r} = (P + \bar{c}\bar{c})\cdot\bar{r} = \bar{r}' + \bar{c}\cos\phi$$

where the vector $\bar{r}'$ lies in the plane Π. Thus

$$J\cdot\bar{c} = \int_T \bar{c}\cdot\bar{r}\bar{r}\, dS = \int_T (\cos\phi)(\bar{r}' + \bar{c}\cos\phi)\, dS$$

$$= \int_{T'} \bar{r}'\, dS' + \bar{c}\int_{T'} \cos\phi\, dS' = \bar{J}_1 + \bar{c}J_2$$

where the meanings of the symbolds $\bar{J}_1$ and J_2 are evident. We first evaluate the scalar J_2.

$$J_2 = \int_{T'} \cos\phi\, dS' = \int_{\psi_1}^{\psi_2}\int_0^R \sqrt{1-\rho^2}\,\rho\, d\rho\, d\psi$$

$$J_2 = \frac{1}{3}\int_{\psi_1}^{\psi_2}[1-(1-R^2)^{3/2}]\,d\psi = \frac{1}{3}\int_{\psi_1}^{\psi_2}\{1 - [(\theta/p)^2-\cos^2\psi]^{-3/2}\sin^3\psi\}d\psi$$

$$= \frac{1}{3}[\psi - \sigma pR(\cos\psi)/\theta^2 - \arccos(\frac{p}{\theta}\cos\psi)]_{\psi_1}^{\psi_2}$$

$$= [e + \sigma(\lambda+\mu)(1-\nu)/\theta^2]/3$$

Before proceding further we establish the unit vector

$$\bar{n} = \bar{c}\times\bar{m} = [(\mu-\nu\lambda)\bar{a} + (\lambda-\mu\nu)\bar{b} - p^2\bar{c}]/p\sigma$$

which lies in plane Π perpendicular to $\bar{m}$. We evidently have

$$\bar{r}' = \rho(\bar{m}\cos\psi + \bar{n}\sin\psi)$$

so that

$$\bar{J}_1 = \int_{\psi_1}^{\psi_2}(\bar{m}\cos\psi + \bar{n}\cos\psi)\int_0^R \rho^2\,d\rho\,d\psi$$

$$= \frac{1}{3}\int_{\psi_1}^{\psi_2}(\bar{m}\cos\psi + \bar{n}\sin\psi)\,R^3\,d\psi$$

$$= \frac{\sigma^3}{3p}\left(\bar{m}\int_{\psi_1}^{\psi_2}\frac{p\cos\psi\,d\psi}{(\sigma^2+p^2\sin^2\psi)^{3/2}} + \bar{n}\int_{\psi_1}^{\psi_2}\frac{p\sin\psi\,d\psi}{(\theta^2-p^2\cos^2\psi)^{3/2}}\right)$$

$$= (\sigma/3\theta^2)[(\bar{m}\theta^2\sin\psi - \bar{n}\sigma^2\cos\psi)/\sqrt{\sigma^2+p^2\sin\psi}]_{\psi_1}^{\psi_2}$$

$$= [\bar{m}\sigma(\lambda-\mu) + \bar{n}(\sigma/\theta)^2(\lambda+\mu)(1-\nu)]/3p$$

$$= [\bar{a}+\bar{b}-(\lambda+\mu)\bar{c}]/3(1+\nu)$$

Thus we have

$$\bar{c}\cdot J = \sigma(\bar{a}+\bar{b})/3(1+\nu) + \bar{c}e/3$$

and

$$J = eU/3 + [\frac{(\bar{a}+\bar{b})\bar{a}\times\bar{b}}{1+\bar{a}\cdot\bar{b}} + (\sim) + (\sim)^2]/3$$

Thus, finally, we arrive at the desired result

$$I_o = (2Mr^2/3)\{U - [(\bar{a}+\bar{b})(\bar{a}\times\bar{b})/(1+\bar{a}\cdot\bar{b}) + (\sim) + (\sim)^2]/2e\}$$

in which we have also accounted for non-unit value of r.

We have removed the original restriction, introduced for convenience, that r = 1, but our analysis thus far has been limited to the case where none of the arcs bounding T exceeds $\pi/2$. We now proceed to remove this restriction. We consider the composition of two triangles, each satis-

the condition (so that the formulas for S, for $\bar{g}$, and for I_o given above are valid) and which, when juxtaposed, form a larger spherical triangle; cf. Figure 3 where C lies on the great circular arc BD. Suppose that the theorems hold for T:ABC and T*:ACD. We shall show that they also hold for $\tilde{T}$:ABD. First consider the areas.

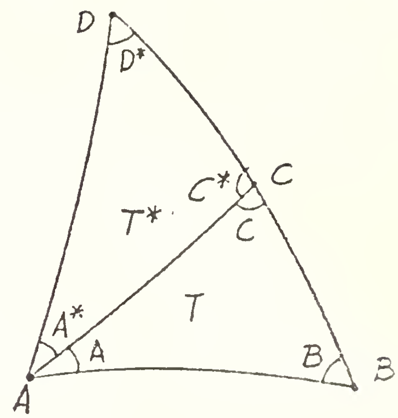

Figure 3

$$S = S+S^* = r^2[(A+B+C-\pi)+(A^*+C^*+D^*-\pi)] = r^2(\tilde{A}+\tilde{B}+\tilde{D}-\pi)$$

since

$$A+A^* = \tilde{A};\quad B = \tilde{B},\ D^* = \tilde{D},\ \text{and } C+C^* = \pi.$$

For the centroid we have

$$2e\tilde{\bar{g}}/r = 2(e\bar{g} + e^*\bar{g}^*)/r = (\bar{v}_{ab} + \bar{v}_{bc} + \bar{v}_{ca}) + (\bar{v}_{ac} + \bar{v}_{cd} + \bar{v}_{da})$$

where

$$\bar{v}_{ab} = \bar{a}\times\bar{b}\ \hat{C}/|\bar{a}\times\bar{b}|, \text{ etc.}$$

Clearly

$$\bar{v}_{ca} + \bar{v}_{ac} = 0$$

and also

$$\bar{v}_{bc} + \bar{v}_{cd} = \bar{v}_{bd}$$

since these vectors are collinear (perpendicular to the plane OBCD) and

$$\text{angle(BOC)} + \text{angle(COD)} = \text{angle(BOD)}$$

Thus

$$2e\tilde{\bar{g}}/r = \bar{v}_{ab} + \bar{v}_{bd} + \bar{v}_{da}$$

and the result is proved.

For the inertia tensor, introduce the notation

$$K_{ab} = (\bar{a}+\bar{b})(\bar{a}\times\bar{b})/(1+\bar{a}\cdot\bar{b}), \text{ etc.}$$

Then

$$\tilde{I}_o = I_o + I_o^*$$

$$= (mr^4/3)[(2Ue-K_{ab}-K_{bc}-K_{ca}) + (2Ue^*-K_{ac}-K_{cd}-K_{da})]$$

$$= (mr^4/3)(2U\bar{e} - K_{ab} - K_{bd} - K_{da} + \mathcal{E})$$

and we will show that

$$\mathcal{E} = K_{bd} - K_{bc} - K_{ca} - K_{ac} - K_{cd}$$

vanishes. Obviously

$$K_{ca} + K_{ac} = 0.$$

Thus, consider

$$K_{bd} - K_{bc} - K_{cd} = \frac{(\bar{b}+\bar{d})(\bar{b}\times\bar{d})}{1+\bar{b}\cdot\bar{d}} - \frac{(\bar{b}+\bar{c})(\bar{b}\times\bar{c})}{1+\bar{b}\cdot\bar{c}} - \frac{(\bar{c}+\bar{d})(\bar{c}\times\bar{d})}{1+\bar{c}\cdot\bar{d}}$$

$$= \left[\frac{(\bar{b}+\bar{d})\sin(BOD)}{1+\cos(BOD)} - \frac{(\bar{b}+\bar{c})\sin(BOC)}{1+\cos(BOC)} - \frac{(\bar{c}+\bar{d})\sin(COD)}{1+\cos(COD)}\right]\bar{e}$$

where $\bar{e}$ is a unit vector perpendicular to the plane OBCD. The vectors appearing in the bracketed expression all lie in this plane and it is not difficult to show that the bracketed expression is zero, so that the result is proved.

That is, if the theorems are true for T and T* and if $\tilde{T} = T + T^*$ is a spherical triangle, then the theorems are true for $\tilde{T}$.

Next, we deal with the case of two spherical triangles T and T* which are complementary on a lune (the figure formed by two intersecting great circles; see Figure 4.) There is no difficulty in showing that

$$S_L = 2r^2C$$

$$S_L\bar{g}_L = \bar{m}\pi r^3\sin(C/2)$$

$$I_{o_L} = (2mr^4/3)[2CU - (\bar{m}\bar{m} - \bar{n}\bar{n})\sin C]$$

where $\bar{m}$, $\bar{n}$, and $\bar{c}$ are mutually perpendicular unit vectors with $\bar{c}$ along the line joining the cusps of the lune and $\bar{m}$ through the center of the lune as shown in Figure 4.

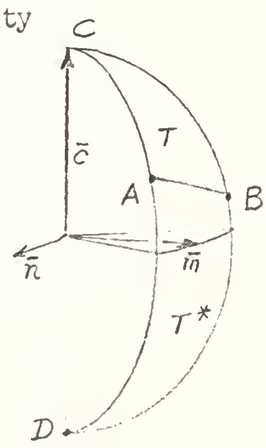

Figure 4

These results are of some interest in themselves. Then, using these results, we can show (but we do not give

the proofs here) that if T and T* are complementary on the lune L, and if the theorems hold for T, then they also hold for T*.

We are now in a position to complete the argument. The three great circle, arcs of which bound the given triangle T, cut the surface of the sphere into eight spherical triangles. There is no difficulty in seeing that there is at least one of these triangles which has two sides whose subtended angles (at the center of the sphere) do not exceed $\pi/2$. Denote this triangle as T_1.

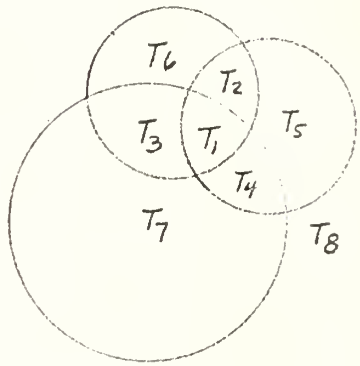

Figure 5

If the third side of T_1 subtends a central angle greater than $\pi/2$, divide T_1 into two subtriangles by an arc of a convenient but arbitrary great circle. At least one of these subtriangles will satisfy the conditions for the theorems.

Thus, the theorems are true for triangle T_1. Pairwise, $T_1 + T_2$, T_1+T_3, T_1+T_4 form lunes. Thus the theorems are true for T_2, T_3, and T_4. Pairwise T_2+T_5, T_3+T_6, and T_4+T_7 form lunes. Thus the theorems are true for T_5, T_6, and T_7. Lastly, T_7+T_8 forms a lune, so that the theorems are true for T_8.

But the given triangle is one of the eight triangles T , ..., T . Thus the theorems are true for the given triangle.

For definiteness, we repeat the statements of the theorems.

$$S = r^2 e = r^2 (A + B + C - \pi)$$

$$\bar{g} = (r/2e)\left[\frac{\bar{a}\times\bar{b}}{|\bar{a}\times\bar{b}|}\hat{C} + \frac{\bar{b}\times\bar{c}}{|\bar{b}\times\bar{c}|}\hat{A} + \frac{\bar{c}\times\bar{a}}{|\bar{c}\times\bar{a}|}\hat{B}\right]$$

$$I_o = (2Mr^2/3)\left\{U - \left[\frac{(\bar{a}+\bar{b})(\bar{a}\times\bar{b})}{1+\bar{a}\cdot\bar{b}} + \frac{(\bar{b}+\bar{c})(\bar{b}\times\bar{c})}{1+\bar{b}\cdot\bar{c}} + \frac{(\bar{c}+\bar{a})(\bar{c}\times\bar{a})}{1+\bar{c}\cdot\bar{a}}\right]/2e\right\}$$

It should be remarked that specification of radius r, unit vectors $\bar{a}$, $\bar{b}$, and $\bar{c}$, and cyclic order (ABC) does not uniquely specify a spherical triangle since there are *two* arcs AB, *two* arcs BC, and *two* arcs CA. Thus, additionally, qualitative information is needed to select the proper case. Vertex angles A, B, and C may be determined by use of this qualitative information and such formulas as

$$\sin^2 A = (\bar{a}\cdot\bar{b}\times\bar{c})^2/[1-(\bar{c}\cdot\bar{a})^2][1-(\bar{a}\cdot\bar{b})^2]$$

Central angles $\hat{A}$, $\hat{B}$, and $\hat{C}$ may be determined by use of this qualitative information and such formulas as

$$\cos \hat{A} = \bar{b}\cdot\bar{c}$$

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